

Chapter Three

Systems of Linear Differential Equations

In this chapter we are going to consider systems of first order ordinary differential equations. These are systems of the form

$$\begin{aligned}x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\&\vdots \\x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t).\end{aligned}$$

Here $x_1(t), \dots, x_n(t)$ are all unknown functions of t and the coefficients a_{ij} are all constant. We can express this system in matrix notation as

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

or

$$\frac{d}{dt} \vec{X}(t) = [A]\vec{X}(t)$$

In addition, there may be an initial condition to satisfy,

$$\begin{bmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$\text{i.e., } \vec{X}(t_0) = \vec{d}$$

Solving these systems will require many of the notions developed in the previous chapter.

3.1 Solution Methods for Linear Systems

Consider the initial value problem for the linear system

$$\begin{aligned}\frac{d}{dt} \vec{X}(t) &= [A]\vec{X}(t) \\ \vec{X}(0) &= \vec{d}\end{aligned} \quad (1.1)$$

Suppose the n by n matrix, A , has real eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding linearly independent eigenvectors $\vec{E}_1, \dots, \vec{E}_n$. Then the general solution to the system of equations is given by

$$\vec{X}(t) = C_1 e^{\lambda_1 t} \vec{E}_1 + \dots + C_n e^{\lambda_n t} \vec{E}_n \quad (1.2)$$

where C_1, \dots, C_n denote arbitrary constants. We will show in a moment how this solution was found but first, let us check that it does solve the equation. For each j , $1 \leq j \leq n$, note that, $\frac{d}{dt}(e^{\lambda_j t}) = \lambda_j e^{\lambda_j t}$ and $A\vec{E}_j = \lambda_j \vec{E}_j$. Then

$$\frac{d}{dt}\vec{X}(t) = C_1 e^{\lambda_1 t} \lambda_1 \vec{E}_1 + \dots + C_n e^{\lambda_n t} \lambda_n \vec{E}_n$$

and

$$\begin{aligned} [A]\vec{X}(t) &= A [C_1 e^{\lambda_1 t} \vec{E}_1 + \dots + C_n e^{\lambda_n t} \vec{E}_n] \\ &= C_1 e^{\lambda_1 t} A \vec{E}_1 + \dots + C_n e^{\lambda_n t} A \vec{E}_n \\ &= C_1 e^{\lambda_1 t} \lambda_1 \vec{E}_1 + \dots + C_n e^{\lambda_n t} \lambda_n \vec{E}_n \end{aligned}$$

Clearly then, the solution (1.2) satisfies equation (1.1). We will now show how this solution is found.

Theorem 1.1 Suppose the n by n matrix A has real eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding linearly independent eigenvectors $\vec{E}_1, \dots, \vec{E}_n$. Then the general solution of (1.1) is given by (1.2)

Proof- Since the eigenvectors are linearly independent, the n by n matrix $P = [\vec{E}_1, \dots, \vec{E}_n]$ whose columns are the eigenvectors, has rank equal to n . Then by theorem 2.2.3 there exists a matrix P^{-1} such that $PP^{-1} = P^{-1}P = I$. In addition,

$$\begin{aligned} AP &= [A\vec{E}_1, \dots, A\vec{E}_n] \\ &= [\lambda_1 \vec{E}_1, \dots, \lambda_n \vec{E}_n] \\ &= PD \end{aligned}$$

where $D = \text{diag}[\lambda_1, \dots, \lambda_n]$ denotes the diagonal matrix whose diagonal entries are the eigenvalues. Now write equation (1) as

$$\frac{d}{dt}\vec{X}(t) = [A]PP^{-1}\vec{X}(t).$$

Then

$$P^{-1} \frac{d}{dt}\vec{X}(t) = P^{-1}[A]PP^{-1}\vec{X}(t).$$

Now P and P^{-1} are constant matrices so

$$P^{-1} \frac{d}{dt}\vec{X}(t) = \frac{d}{dt}(P^{-1}\vec{X}(t))$$

and if we let $\vec{Y}(t) = P^{-1}\vec{X}(t)$, then our equation reduces to

$$\frac{d}{dt}\vec{Y}(t) = P^{-1}[A]P\vec{Y}(t).$$

But $AP = PD$ and so $P^{-1}[A]P = P^{-1}PD = D$ and our equation reduces further to

$$\frac{d}{dt}\vec{Y}(t) = D\vec{Y}(t).$$

That is,

$$\frac{d}{dt} \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1(t) \\ \vdots \\ \lambda_n y_n(t) \end{bmatrix}$$

This system is uncoupled since each equation in the system contains a single unknown function. The uncoupled system is easily solved to find

$$\vec{Y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} C_1 e^{\lambda_1 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{bmatrix}.$$

Then since $\vec{Y}(t) = P^{-1}\vec{X}(t)$,

$$\begin{aligned} \vec{X}(t) &= P\vec{Y}(t) \\ &= y_1(t)\vec{E}_1 + \dots + y_n(t)\vec{E}_n \\ &= C_1 e^{\lambda_1 t}\vec{E}_1 + \dots + C_n e^{\lambda_n t}\vec{E}_n \end{aligned}$$

which is what we were supposed to prove. ■

Corollary 1.1- The unique solution to the initial value problem (1) is found by choosing the arbitrary constants C_i to be the unique solution of the system of algebraic equations $P\vec{C} = \vec{d}$, where $\vec{C} = [C_1, \dots, C_n]^T$. If the eigenvectors are mutually orthogonal then the arbitrary constants are given by

$$C_i = \frac{\vec{d} \cdot \vec{E}_i}{\vec{E}_i \cdot \vec{E}_i} \quad i = 1, \dots, n$$

Proof- It follows from (2) that $\vec{X}(0) = P\vec{Y}(0) = \vec{d}$ if $C_1\vec{E}_1 + \dots + C_n\vec{E}_n = \vec{d}$. But recalling section 2.2.4, we see $P\vec{C} = C_1\vec{E}_1 + \dots + C_n\vec{E}_n$.

If the eigenvectors are mutually orthogonal then the arbitrary constants can be found more easily by noting that if

$$C_1\vec{E}_1 + \dots + C_n\vec{E}_n = \vec{d}$$

then for any i $1 \leq i \leq n$,

$$\begin{aligned} \vec{d} \cdot \vec{E}_i &= (C_1\vec{E}_1 + \dots + C_n\vec{E}_n) \cdot \vec{E}_i \\ &= C_1\vec{E}_1 \cdot \vec{E}_i + \dots + C_n\vec{E}_n \cdot \vec{E}_i \end{aligned}$$

But $\vec{E}_i \cdot \vec{E}_j = 0$ if $i \neq j$ so

$$C_1\vec{E}_1 \cdot \vec{E}_i + \dots + C_n\vec{E}_n \cdot \vec{E}_i = C_i\vec{E}_i \cdot \vec{E}_i$$

Since the \vec{E}_i 's are eigenvectors $\vec{E}_i \cdot \vec{E}_i \neq 0$ and the result is proved. ■

For each i , $1 \leq i \leq n$, $e^{\lambda_i t}\vec{E}_i$ is a vector valued function of t that satisfies the system (1). These n functions are said to be linearly dependent if there exist constants C_1, \dots, C_n , not all zero, such that $C_1 e^{\lambda_1 t}\vec{E}_1 + \dots + C_n e^{\lambda_n t}\vec{E}_n = \vec{0}$ for all t . If the n functions are not linearly dependent, they are said to be linearly independent. Since the vectors $\vec{E}_1, \dots, \vec{E}_n$ are linearly independent vectors, it is clear that $e^{\lambda_1 t}\vec{E}_1, \dots, e^{\lambda_n t}\vec{E}_n$ are linearly independent as functions. Thus the system (1) has n linearly independent solutions.

Examples

1. To illustrate the use of theorem 1.1, we are going to consider the diffusion of a contaminant in a 1-dimensional medium. Assume the medium is a long channel in which the contaminant can travel from left to right or right to left but there is no diffusion in the the transverse direction. We will think of the channel as being divided into a number of cells

with the cells being separated from one another by faces. The cells are numbered, $1 \leq k \leq n$, with cell k to the right of cell $k - 1$ and to the left of cell $k + 1$. The faces bounding cell k are numbered as face k on the left of cell k and face $k + 1$ on the right side of cell k . If we let $u_k(t)$ denote the concentration level in cell k at time t , then conservation of contaminant is expressed as follows,

$$\frac{du_k(t)}{dt} = [\text{flow in} - \text{flow out}]$$

where flow in and flow out are the rates of mass flow through face k and face $k + 1$, respectively. If we assume that F_k , the mass flow rate across face k is proportional to the concentration gradient across face k , then we can write

$$F_k = D_k(u_k(t) - u_{k-1}(t))$$

where D_k denotes the material dependent proportionality constant known as the coefficient of diffusion. Then we have the following equation for the concentration in cell k

$$\begin{aligned} \frac{du_k(t)}{dt} &= D_k(u_{k-1}(t) - u_k(t)) - D_{k+1}(u_k(t) - u_{k+1}(t)) \\ &= D_k u_{k-1}(t) - [D_k + D_{k+1}]u_k(t) + D_{k+1}u_{k+1}(t) \end{aligned}$$

If the channel is comprised of n cells then there are n ODE's for the n unknown concentrations $u_n(t)$. For simplicity of the example, let us suppose $n = 3$ and that the ends of the channel are sealed so the contaminant can move within the channel but it can not escape at the ends. This amounts to assuming that $D_1 = D_4 = 0$. Then the equations we must solve are as follows

$$\begin{aligned} \frac{du_1(t)}{dt} &= -D_2u_1 + D_2u_2 \\ \frac{du_2(t)}{dt} &= D_2u_1 - [D_2 + D_3]u_2 + D_3u_3 \\ \frac{du_3(t)}{dt} &= D_3u_2 - D_3u_3 \end{aligned}$$

If we make the additional assumption that $D_2 = D_3 = 1$, then the system of equations in matrix notation, becomes

$$\frac{d}{dt} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}.$$

Finally, let us assume that in the initial state, the concentration in cell 3 is 1 (saturated) and the concentration in cells 1 and 2 is zero (no contaminant). Then

$$\begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In order to find the general solution of this system, we first find the eigenvalues and eigenvectors for the coefficient matrix. Proceeding as we did in the previous chapter, we find

$$\lambda_1 = 0 \quad \vec{E}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = -1 \quad \vec{E}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad \lambda_3 = -3 \quad \vec{E}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Since the coefficient matrix is symmetric, the eigenvalues are all real and the eigenvectors are mutually orthogonal. Now theorem 1.1 asserts that the general solution to this system is

$$\begin{aligned} \vec{u}(t) &= C_1 e^{\lambda_1 t} \vec{E}_1 + C_2 e^{\lambda_2 t} \vec{E}_2 + C_3 e^{\lambda_3 t} \vec{E}_3 \\ &= C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \end{aligned}$$

We must now find values for C_1, C_2 and C_3 so that the initial condition is satisfied. That is,

$$C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In general we would have to solve this system of algebraic equations in order to determine the C_i 's but since the eigenvectors are mutually orthogonal, we have

$$\begin{aligned} C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \text{or} \quad 3C_1 &= 1 \end{aligned}$$

Similarly, we find

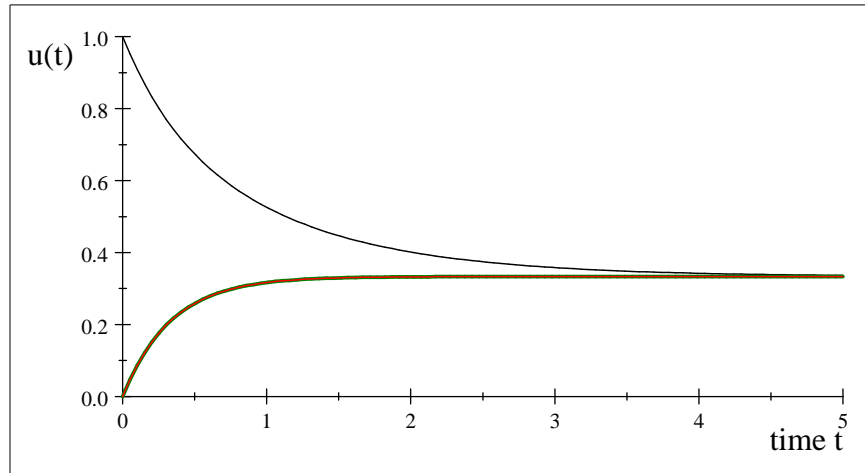
$$\begin{aligned} 2C_2 &= 1 \\ \text{and} \quad 6C_3 &= 1 \end{aligned}$$

Then the unique solution of the initial value problem is

$$\begin{aligned} \vec{u}(t) &= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{6} e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} - \frac{1}{2} e^{-t} + \frac{1}{6} e^{-3t} \\ \frac{1}{3} - \frac{1}{3} e^{-3t} \\ \frac{1}{3} + \frac{1}{2} e^{-t} + \frac{1}{6} e^{-3t} \end{bmatrix} \end{aligned}$$

If we plot the concentrations in the three cells versus t for $0 \leq t \leq 5$, we see that the concentrations in cells 1 and 2 are nearly indistinguishable from each other and that the concentration in cell 3 drops from 1 to $1/3$ while the concentrations in the other two cells increase to $1/3$. That is, the concentration becomes the same in all three cells, which is

what we would expect to happen if the ends of the channel are sealed.



Channel with sealed ends

If we assume $D_k = 1$ for all five faces and the concentration to the left of cell 1 and to the right of cell 3 are equal to zero, this amounts to supposing that the contaminant can escape from the channel and that both ends of the channel are contaminant free. In this case the coefficient matrix changes to

$$\begin{bmatrix} -(D_1 + D_2) & D_2 & 0 \\ D_2 & -(D_2 + D_3) & D_3 \\ 0 & D_3 & -(D_3 + D_4) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

and now the eigenvalues and eigenvectors are

$$\lambda_1 = \sqrt{2} - 2 \quad \vec{E}_1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2 \quad \vec{E}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -\sqrt{2} - 2 \quad \vec{E}_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

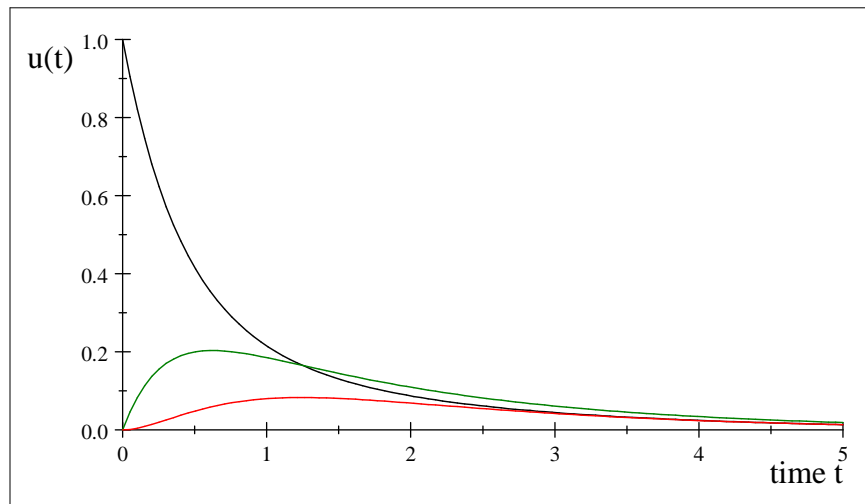
The general solution of the system is now equal to

$$\vec{u}(t) = C_1 e^{(\sqrt{2}-2)t} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{(-\sqrt{2}-2)t} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

The unique solution to the initial value problem now becomes

$$\begin{aligned} \vec{u}(t) &= \frac{1}{4} e^{(\sqrt{2}-2)t} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} + \frac{1}{2} e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{4} e^{(-\sqrt{2}-2)t} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} e^{(\sqrt{2}-2)t} - \frac{1}{2} e^{-2t} + \frac{1}{4} e^{(-\sqrt{2}-2)t} \\ \sqrt{2} \left[\frac{1}{4} e^{(\sqrt{2}-2)t} - \frac{1}{4} e^{(-\sqrt{2}-2)t} \right] \\ \frac{1}{4} e^{(\sqrt{2}-2)t} + \frac{1}{2} e^{-2t} + \frac{1}{4} e^{(-\sqrt{2}-2)t} \end{bmatrix} \end{aligned}$$

We can see from the plot that now the concentration in all three cells tends to zero as t tends to infinity, which means that all the contaminant eventually leaves the channel. Note that as u_3 decreases, u_2 at first increases, followed by u_1 . This is because the contaminant flowing out of cell 3 flows first into cell 2 and then from cell 2 into cell 1. Eventually, all the contaminant flows out of the channel.



Channel with open ends

Notice that in both of these two examples the coefficient matrices were symmetric. This was not a coincidence. When the differential equations in the model are derived from a conservation principle (in this case it is the conservation of contaminant) the coefficient matrix will be symmetric. Since many of the mathematical models leading to systems of linear ODE's are derived from conservation laws, it is not unusual to have symmetric coefficient matrices and correspondingly mutually orthogonal eigenvectors.

2. Let us now consider an example of an initial value problem where the coefficient matrix is not symmetric; e.g.

$$\frac{d}{dt} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix},$$

$$\begin{bmatrix} u_1(0) & u_2(0) & u_3(0) \end{bmatrix}^\top = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^\top$$

Proceeding as in the previous chapter, we find

$$\lambda_1 = -2, \vec{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \lambda_2 = -3, \vec{E}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = -4, \vec{E}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and since the matrix is not symmetric but the eigenvalues are distinct, the eigenvectors are linearly independent but not mutually orthogonal. The general solution to the system is

$$\begin{aligned} \vec{u}(t) &= C_1 e^{\lambda_1 t} \vec{E}_1 + C_2 e^{\lambda_2 t} \vec{E}_2 + C_3 e^{\lambda_3 t} \vec{E}_3 \\ &= C_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{-4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

If we assume the same initial condition as in the previous examples, then in order to find the values for C_i , we have to solve the following system of algebraic equations

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Then $C_2 = C_3 = \frac{1}{2}$ and $C_1 = 1$. The unique solution to the initial value problem is then

$$\begin{aligned} \vec{u}(t) &= e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} e^{-3t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} - \frac{1}{2} e^{-3t} + \frac{1}{2} e^{-4t} \\ \frac{1}{2} e^{-3t} \\ \frac{1}{2} e^{-3t} + \frac{1}{2} e^{-4t} \end{bmatrix} \end{aligned}$$

Exercises

Let $A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

1. Solve $\frac{d}{dt}\vec{X}(t) = A_1\vec{X}(t)$ $\vec{X}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

2. Solve $\frac{d}{dt}\vec{X}(t) = A_2\vec{X}(t)$ $\vec{X}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

3. Solve $\frac{d}{dt}\vec{X}(t) = A_3\vec{X}(t)$ $\vec{X}(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

4. Solve $\frac{d}{dt}\vec{X}(t) = A_4\vec{X}(t)$ $\vec{X}(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

3.2 Complex Eigenvalues

In the previous section, we have considered first order systems or linear ODE's where the coefficient matrix had real eigenvalues. In this section we will see how to modify the approach slightly in order to accommodate matrices with complex eigenvalues. We could treat the case of complex eigenvalues and eigenvectors exactly as we handled the real valued case but then the solutions to the system of ODE's would be complex valued. In most applications, we would prefer to have real valued solutions and in order to achieve this we will proceed as follows. If the matrix A has a complex eigenvalue $\lambda = \alpha + i\beta$, with corresponding eigenvector $\vec{E} = \vec{u} + i\vec{v}$, (\vec{E} will then have complex entries), then the conjugate of λ will also be an eigenvalue for A and the corresponding eigenvector will be the conjugate of \vec{E} . The conjugate eigenvalue and eigenvector will be denoted by $\bar{\lambda} = \alpha - i\beta$, and $\vec{E}^* = \vec{u} - i\vec{v}$, respectively. Now $e^{\lambda t}\vec{E}$ and $e^{\bar{\lambda}t}\vec{E}^*$ are two linearly independent solutions to the original system. However, recall that, $e^{i\beta t} = \cos \beta t + i \sin \beta t$ and hence,

$$\begin{aligned} e^{\lambda t}\vec{E} &= e^{(\alpha+i\beta)t}[\vec{u} + i\vec{v}] \\ &= e^{\alpha t}[\cos \beta t + i \sin \beta t][\vec{u} + i\vec{v}] \\ &= e^{\alpha t}[\cos \beta t \vec{u} - \sin \beta t \vec{v}] + ie^{\alpha t}[\cos \beta t \vec{v} + \sin \beta t \vec{u}] \\ &= \vec{U}(t) + i\vec{V}(t) \end{aligned}$$

It is clear that both $\vec{U}(t)$ and $\vec{V}(t)$ are solutions to the original system since

$$\frac{d}{dt}(e^{\lambda t}\vec{E}) = \frac{d}{dt}\vec{U}(t) + i\frac{d}{dt}\vec{V}(t)$$

$$A(e^{\lambda t}\vec{E}) = A\vec{U}(t) + iA\vec{V}(t)$$

$$\text{so } \frac{d}{dt}\vec{U}(t) + i\frac{d}{dt}\vec{V}(t) = A\vec{U}(t) + iA\vec{V}(t)$$

Moreover, $\vec{U}(t)$ and $\vec{V}(t)$ are real valued and linearly independent. Thus, if we prefer to have real valued solutions to the system of ODE's, we can replace the complex valued solutions $e^{\lambda t}\vec{E}$ and $e^{\bar{\lambda}t}\vec{E}^*$ with the real valued solutions $\vec{U}(t)$ and $\vec{V}(t)$.

Examples

1. Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The eigenvalues for A are $\lambda = 1 \pm i$. To find the eigenvector \vec{E} corresponding to $\lambda = 1 + i$, we write

$$A - (1 + i)I = \begin{bmatrix} -1 - i & 1 \\ -2 & 1 - i \end{bmatrix}$$

which implies that $\vec{E} = [e_1, e_2]^T$ with $(-1 - i)e_1 + e_2 = 0$. That is, $e_2 = (1 + i)e_1$ and $\vec{E} = [1, 1 + i]^T$. Note that there was no need to employ the second equation in $(A - \lambda I)\vec{E} = \vec{0}$, since it must contain the same information as the first equation. Now a complex valued solution to the system is given by

$$\vec{X}(t) = e^{(1+i)t}\begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$$

If we prefer to have real valued solutions we note that

$$\begin{aligned} \vec{X}(t) &= e^t[\cos t + i\sin t]\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= e^t\left(\cos t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + ie^t\left(\cos t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ &= e^t\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + ie^t\begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix} \end{aligned}$$

Hence

$$\vec{U}_1(t) = e^t\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} \text{ and } \vec{U}_2(t) = e^t\begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

are two linearly independent real valued solutions for the system. The general real valued solution is then given by

$$\vec{X}(t) = C_1 e^t \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

2. Consider

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

In one of the examples of the previous chapter, we found the real eigenpair $\{\lambda_1 = 4, \vec{E}_1 = [0, 0, 1]^T\}$ and complex eigenvalues $\lambda = 1 \pm 3i$ with complex eigenvectors $\vec{E}_\pm = [1, \pm i, 0]^T$. Then

$$\vec{X}(t) = e^{(1-3i)t} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

is a complex valued solution for the system. A corresponding pair of real valued solutions is found from

$$\begin{aligned} \vec{X}(t) &= e^t [\cos 3t + i \sin 3t] \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \\ &= e^t \left(\cos 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) + i e^t \left(\sin 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \cos 3t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \\ &= e^t \begin{bmatrix} \cos 3t \\ \sin 3t \\ 0 \end{bmatrix} + i e^t \begin{bmatrix} \sin 3t \\ -\cos 3t \\ 0 \end{bmatrix} \end{aligned}$$

That is,

$$\vec{U}_1(t) = e^t \begin{bmatrix} \cos 3t \\ \sin 3t \\ 0 \end{bmatrix} \text{ and } \vec{U}_2(t) = e^t \begin{bmatrix} \sin 3t \\ -\cos 3t \\ 0 \end{bmatrix}$$

are two linearly independent real valued solutions for the system. The general solution is

$$\begin{aligned}\vec{X}(t) &= C_1 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} \cos 3t \\ \sin 3t \\ 0 \end{bmatrix} + C_3 e^t \begin{bmatrix} \sin 3t \\ -\cos 3t \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} C_2 e^t \cos 3t + C_3 e^t \sin 3t \\ C_2 e^t \sin 3t - C_3 e^t \cos 3t \\ C_1 e^{4t} \end{bmatrix}\end{aligned}$$

Exercises

In each problem, find the real valued general solution and then find a particular solution which satisfies the initial condition.

$$1. \quad \frac{d}{dt} \vec{X}(t) = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \vec{X}(t) \quad \vec{X}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$2. \quad \frac{d}{dt} \vec{X}(t) = \begin{bmatrix} 4 & 2 \\ -5 & 2 \end{bmatrix} \vec{X}(t) \quad \vec{X}(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$3. \quad \frac{d}{dt} \vec{X}(t) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \vec{X}(t) \quad \vec{X}(0) = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

$$4. \quad \frac{d}{dt} \vec{X}(t) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -3 & 2 \end{bmatrix} \vec{X}(t) \quad \vec{X}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$5. \quad \frac{d}{dt} \vec{X}(t) = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix} \vec{X}(t) \quad \vec{X}(0) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

3.3 Exponential of an $n \times n$ matrix A

In the first section of the first chapter, we learned that the solution of the scalar initial value problem, $x'(t) = Ax(t)$, $x(0) = x_0$, where A denotes a real number, is $x(t) = e^{At}x_0$ for any constant x_0 . We will learn in this section that this is also the solution when $x(t)$ and x_0 are vectors in R^n and A is an n by n matrix. But first we must explain how to interpret e^{At} when A is an n by n matrix.

For any real number x , the exponential function e^x can be defined as follows

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad (3.1)$$

For an $n \times n$ matrix A , we can define

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \quad (3.2)$$

In the special case that A is a diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_n \end{bmatrix} = \text{diag}[\lambda_1, \dots, \lambda_n]$$

then (3.2) implies that

$$e^A = \begin{bmatrix} e^{\lambda_1} & 0 \\ & \ddots \\ 0 & e^{\lambda_n} \end{bmatrix} = \text{diag}[e^{\lambda_1}, \dots, e^{\lambda_n}]$$

We must collect a few facts about the exponential of a matrix.

- For any real numbers a and b , $e^{a+b} = e^a e^b$ but for $n \times n$ matrices A and B

$$e^{A+B} = e^A e^B \quad \text{if and only if} \quad AB = BA$$

- For any $n \times n$ matrix A , e^A is invertible and the inverse is given by $(e^A)^{-1} = e^{-A} = I - A + \frac{1}{2!}A^2 - \frac{1}{3!}A^3 + \dots$ (i.e., e^A is invertible for any A).
- If I denotes the $n \times n$ identity matrix, then $e^{tI} = e^t I$, and for any $n \times n$ matrix A , $e^{0A} = I$.

Now we can state and prove,

Theorem 3.1- For any $n \times n$ matrix A , the unique solution of the initial value problem

$$\vec{X}'(t) = A\vec{X}(t), \quad \vec{X}(0) = \vec{v}$$

is given by $\vec{X}(t) = e^{tA}\vec{v}$.

Proof-

$$\begin{aligned} e^{tA}\vec{v} &= \vec{v} + tA\vec{v} + \frac{t^2}{2!}A^2\vec{v} + \frac{t^3}{3!}A^3\vec{v} + \dots \\ \frac{d}{dt}(e^{tA}\vec{v}) &= 0 + A\vec{v} + tA^2\vec{v} + \frac{t^2}{2!}A^3\vec{v} + \dots \\ &= A \left[\vec{v} + tA\vec{v} + \frac{t^2}{2!}A^2\vec{v} + \frac{t^3}{3!}A^3\vec{v} + \dots \right] \\ &= A[e^{tA}\vec{v}] \end{aligned}$$

Finally, $\vec{X}(0) = e^{0A}\vec{v} = I\vec{v} = \vec{v}$. ■

Note the following consequences of (3.2):

$$\begin{aligned} \text{If } A\vec{v} &= \vec{0} \text{ then } e^{tA}\vec{v} = \vec{v} \\ \text{If } A^2\vec{v} &= \vec{0} \text{ then } e^{tA}\vec{v} = \vec{v} + tA\vec{v} \\ \text{If } A^3\vec{v} &= \vec{0} \text{ then } e^{tA}\vec{v} = \vec{v} + tA\vec{v} + \frac{t^2}{2!}A^2\vec{v} \end{aligned}$$

Now suppose \vec{v} is an eigenvector for A corresponding to the eigenvalue λ ; i.e. $(A - \lambda I)\vec{v} = \vec{0}$. Then

$$e^{tA}\vec{v} = e^{t(A-\lambda I)}e^{t\lambda I}\vec{v} = e^{t\lambda I}e^{t(A-\lambda I)}\vec{v} = e^{\lambda t}e^{t(A-\lambda I)}\vec{v}$$

But $(A - \lambda I)\vec{v} = \vec{0}$ and it follows that $e^{t(A-\lambda I)}\vec{v} = \vec{v}$. That is

$$\text{If } (A - \lambda I)\vec{v} = \vec{0} \text{ then } e^{tA}\vec{v} = e^{\lambda t}\vec{v}. \quad (3.3)$$

Next, suppose \vec{v} is an eigenvector for A corresponding to the eigenvalue λ , and \vec{w} satisfies $(A - \lambda I)\vec{w} = \vec{v}$. Then $(A - \lambda I)^2\vec{w} = (A - \lambda I)\vec{v} = \vec{0}$. In that case,

$$\begin{aligned} e^{tA}\vec{w} &= e^{t\lambda I}e^{t(A-\lambda I)}\vec{w} \\ &= e^{t\lambda I}e^{t(A-\lambda I)}\vec{w} = e^{\lambda t}e^{t(A-\lambda I)}\vec{w} \\ &= e^{\lambda t}[\vec{w} + t(A - \lambda I)\vec{w}] \\ &= e^{\lambda t}[\vec{w} + t\vec{v}] \end{aligned}$$

That is,

$$\text{If } (A - \lambda I)\vec{w} = \vec{v} \text{ then } e^{tA}\vec{w} = e^{\lambda t}[\vec{w} + t\vec{v}] \quad (3.4)$$

Finally, if \vec{v} and \vec{w} are as above, and \vec{u} satisfies $(A - \lambda I)\vec{u} = \vec{w}$ then $(A - \lambda I)^3\vec{u} = (A - \lambda I)^2\vec{w} = (A - \lambda I)\vec{v} = \vec{0}$ and in this case

$$e^{tA}\vec{u} = e^{t\lambda I}e^{t(A-\lambda I)}\vec{u} = e^{\lambda t}\left[\vec{u} + t\vec{w} + \frac{t^2}{2}\vec{v}\right]$$

so

$$\text{If } (A - \lambda I)\vec{u} = \vec{w} \text{ then } e^{tA}\vec{u} = e^{\lambda t}\left[\vec{u} + t\vec{w} + \frac{t^2}{2}\vec{v}\right] \quad (3.5)$$

The results (3.3) to (3.5) will now be used in order to find the solution for a system of first order linear ODE's when the matrix A does not have a full set of eigenvectors.

3.4 Repeated Eigenvalues

The matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

is immediately seen to have eigenvalues $\lambda = 2, 2, 3$. We say that the eigenvalue $\lambda = 2$ has algebraic multiplicity 2 and $\lambda = 3$ has algebraic multiplicity 1. Often, but not always, a repeated eigenvalue like $\lambda = 2$ can pose difficulties in solving the system of linear differential equations having A as its coefficient matrix. In this example, the eigenvectors associated with $\lambda = 2$ are found as follows:

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

This tells us that $x_3 = 0$, while x_1 and x_2 are free so the eigenvector associated with $\lambda = 2$ is given by

$$\vec{E} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Evidently, there are two independent eigenvectors associated with $\lambda = 2$,

$$\vec{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{E}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then the geometric multiplicity of $\lambda = 2$, the number of independent eigenvectors, is the same as the algebraic multiplicity. For $\lambda = 3$ we find

$$\vec{E}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Then the general solution of the linear system $\vec{X}'(t) = A\vec{X}(t)$ is

$$\vec{X}(t) = C_1 e^{2t} \vec{E}_1 + C_2 e^{2t} \vec{E}_2 + C_3 e^{3t} \vec{E}_3$$

Evidently, in this case that the fact that the eigenvalue $\lambda = 2$ is repeated does not pose a problem for solving the associated linear system of differential equations. On the other hand, consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

whose eigenvalues are found to be $\lambda = 2, 2$ but since

$$A - 2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

there is only the single eigenvector

$$\vec{E}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

we see that the algebraic multiplicity is 2 while the geometric multiplicity is just 1. In order to solve the associated system of linear differential equations, we need another independent

vector. For this purpose, we look for \vec{E}_2 satisfying

$$(A - 2I)\vec{E}_2 = \vec{E}_1$$

Then, since \vec{E}_1 is an eigenvector for $\lambda = 2$, $(A - 2I)\vec{E}_1 = 0$ and,

$$(A - 2I)^2\vec{E}_2 = (A - 2I)\vec{E}_1 = 0$$

But

$$(A - 2I)^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so any choice of \vec{E}_2 satisfies this equation. A simple choice for \vec{E}_2 would be,

$$\vec{E}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then, according to (3.3) and (3.4), two independent solutions to $\vec{X}'(t) = A\vec{X}(t)$ are given by

$$\vec{X}_1(t) = e^{tA}\vec{E}_1 = e^{2t}\vec{E}_1$$

$$\vec{X}_2(t) = e^{tA}\vec{E}_2 = e^{2t}(\vec{E}_2 + t\vec{E}_1)$$

and the general solution to the system is

$$\vec{X}(t) = C_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} (1+t)e^{2t} \\ te^{2t} \end{bmatrix}$$

We refer to \vec{E}_2 as a generalized eigenvector for A .

Additional Examples

1. Consider $\vec{X}'(t) = A\vec{X}(t)$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Since A is triangular, the eigenvalues are seen to be $\lambda_1 = 3$ and $\lambda_2 = \lambda_3 = 1$. We find the eigenvector $\vec{E}_1 = [1, 2, 2]^T$ corresponding to $\lambda_1 = 3$ and $\vec{E}_2 = [1, 0, 0]^T$ corresponding to $\lambda_2 = \lambda_3 = 1$. An additional independent vector is needed in order to form the general solution to $\vec{X}'(t) = A\vec{X}(t)$.

If we suppose \vec{E}_3 satisfies

$$(A - I)\vec{E}_3 = \vec{E}_2$$

then

$$(A - I)^2\vec{E}_3 = (A - I)\vec{E}_2 = \vec{0}.$$

Note that

$$(A - I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

and it follows that $(A - I)^2 \vec{E}_3 = \vec{0}$ if

$$\vec{E}_3 = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

for any choice of x_1 and x_2 as long as $\vec{E}_3 \neq \vec{E}_2$. If we choose $\vec{E}_3 = [0, 1, 0]^T$ then $\vec{X}_2(t) = e^{tA} \vec{E}_3 = e^t(\vec{E}_3 + t\vec{E}_2)$ and

$$\vec{X}_1(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{X}_2(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{X}_3(t) = e^t \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

are three linearly independent solutions to $\vec{X}'(t) = A \vec{X}(t)$.

2. Consider $\vec{X}'(t) = A \vec{X}(t)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

This matrix has eigenvalue, $\lambda = 1$ with algebraic multiplicity 3. The geometric multiplicity is 1 and the only eigenvector is $\vec{E}_1 = [-2, 0, 1]^T$. In order to find two more generalized eigenvectors, we first look for \vec{E}_2 satisfying

$$(A - I)\vec{E}_2 = \vec{E}_1$$

so that $(A - I)^2 \vec{E}_2 = \vec{0}$.

Since

$$(A - I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

we see that $\vec{E}_2 = [x_1, x_2, x_3]^T$ must have $x_1 + 2x_3 = 0$ and x_2 is arbitrary. The simplest choice for \vec{E}_2 is $\vec{E}_2 = [0, 1, 0]^T$. To find another generalized eigenvector, we look for \vec{E}_3 satisfying

$$(A - I)\vec{E}_3 = \vec{E}_2$$

so that $(A - I)^3 \vec{E}_3 =$
 $(A - I)^2 \vec{E}_2 = \vec{0}$.

In this case, $(A - I)^3$ is the zero matrix (all zeroes) so any choice for \vec{E}_3 will work as long as it is independent of \vec{E}_2 and \vec{E}_1 . The simplest choice is $\vec{E}_3 = [1, 0, 0]^T$. Since we have chosen \vec{E}_2 and \vec{E}_3 such that

$$(A - I)^2 \vec{E}_2 = \vec{0}$$

and $(A - I)^3 \vec{E}_3 = \vec{0}$

we have three independent solutions to $\vec{X}'(t) = A \vec{X}(t)$. They are

$$\begin{aligned}\vec{X}_1(t) &= e^{tA} \vec{E}_1 = e^t \vec{E}_1 \\ \vec{X}_2(t) &= e^{tA} \vec{E}_2 = e^t (\vec{E}_2 + t \vec{E}_1) \\ \vec{X}_3(t) &= e^{tA} \vec{E}_3 = e^t (\vec{E}_3 + t \vec{E}_2 + \frac{1}{2} t^2 \vec{E}_1)\end{aligned}$$

and the general solution for the system is

$$\vec{X}(t) = C_1 e^t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} -2t \\ 1 \\ t \end{bmatrix} + C_3 e^t \begin{bmatrix} 1 - t^2 \\ t \\ \frac{1}{2} t^2 \end{bmatrix}$$

Exercises

For each of the following matrices, find the eigenvalues and find the algebraic and geometric multiplicity of each eigenvalue. Then find the general solution to $\vec{X}'(t) = A \vec{X}(t)$.

$$\begin{aligned}A_1 &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} & A_2 &= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} & A_3 &= \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ A_4 &= \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} & A_5 &= \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} & A_6 &= \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}\end{aligned}$$

5. The Inhomogeneous Equation

The inhomogeneous version of the initial value problem (1.1) is the following

$$\begin{aligned}\frac{d}{dt} \vec{X}(t) &= [A] \vec{X}(t) + \vec{F}(t) & (5.1) \\ \vec{X}(0) &= \vec{d}\end{aligned}$$

We already know the solution to the homogeneous equation, it is

$$\vec{X}_H(t) = C_1 e^{\lambda_1 t} \vec{E}_1 + \dots + C_n e^{\lambda_n t} \vec{E}_n \quad (5.2)$$

where $\{\lambda_i, \vec{E}_i : 1 \leq i \leq n\}$ denote the eigenpairs for A . If we apply the method of variation of parameters that was discussed in chapter 1, then we assume the particular solution for (5.1) is of the form

$$\vec{X}_p(t) = C_1(t) e^{\lambda_1 t} \vec{E}_1 + \dots + C_n(t) e^{\lambda_n t} \vec{E}_n \quad (5.3).$$

If we substitute (5.3) into (5.1) then we find

$$C'_1(t) e^{\lambda_1 t} \vec{E}_1 + \dots + C'_n(t) e^{\lambda_n t} \vec{E}_n = \vec{F}(t). \quad (5.4)$$

This is the analogue for systems of what we encountered in chapter 1 for a single equation.

We can write (5.4) in matrix notation as follows

$$\begin{bmatrix} e^{\lambda_1 t} \vec{E}_1, \dots, e^{\lambda_n t} \vec{E}_n \end{bmatrix} \begin{bmatrix} C_1'(t) \\ \vdots \\ C_n'(t) \end{bmatrix} = \vec{F}(t).$$

Here $[Y(t)] = [e^{\lambda_1 t} \vec{E}_1, \dots, e^{\lambda_n t} \vec{E}_n]$ is the matrix whose j -th column is the j -th component of the general homogeneous solution, $\vec{X}_j(t) = e^{\lambda_j t} \vec{E}_j$. We refer to $[Y(t)]$ as the fundamental matrix for (5.1). Since the columns are independent, the fundamental matrix is invertible, hence

$$\begin{bmatrix} C_1'(t) \\ \vdots \\ C_n'(t) \end{bmatrix} = [Y(t)]^{-1} \vec{F}(t)$$

and

$$\begin{bmatrix} C_1(t) \\ \vdots \\ C_n(t) \end{bmatrix} = \int_0^t [Y(s)]^{-1} \vec{F}(s) ds$$

Now (5.3) implies

$$\vec{X}_p(t) = [Y(t)] \vec{C}(t) = [Y(t)] \int_0^t [Y(s)]^{-1} \vec{F}(s) ds \quad (5.5)$$

We will illustrate with an example.

Consider the system

$$\vec{X}'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{X}(t) + \begin{bmatrix} t \\ 1 \end{bmatrix}$$

The eigenpairs for this matrix are $\left\{ \vec{E}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda_1 = 1, \vec{E}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 = 2 \right\}$ and

$$Y(t) = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

We recall now that there is a simple formula for the inverse of a 2 by 2 matrix; i.e.,

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\det A = ad - bc$. In this case, we have

$$Y(t)^{-1} = e^{-3t} \begin{bmatrix} e^{2t} & -e^{2t} \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ 0 & e^{-2t} \end{bmatrix}$$

and

$$\int_0^t [Y(s)]^{-1} \vec{F}(s) ds = \int_0^t \begin{bmatrix} se^{-s} - e^{-s} \\ e^{-2s} \end{bmatrix} ds = \begin{bmatrix} -te^{-t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix}$$

Then

$$\begin{aligned} \vec{X}_p(t) &= [Y(t)]\vec{C}(t) = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -te^{-t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} -t + \frac{1}{2}e^{2t} - \frac{1}{2} \\ \frac{1}{2}e^{2t} - \frac{1}{2} \end{bmatrix} \end{aligned}$$

This is a particular solution for the inhomogeneous system. Note that $\vec{X}_p(0) = \vec{0}$ so, if there were initial conditions to be satisfied, these could be accommodated by choosing the constants the homogeneous solution.

It may be worth noting that it is possible to avoid dealing with the fundamental matrix and its inverse. Consider the simpler case where the eigenvectors of A are mutually orthogonal. Then it follows from (5.4) that for each j ,

$$C_j'(t) = \mu_j e^{-\lambda_j t} \vec{E}_j \cdot \vec{F}(t)$$

where $\mu_j = (\vec{E}_j \cdot \vec{E}_j)^{-1}$. Then

$$C_j(t) = \mu_j \int_0^t e^{-\lambda_j s} (\vec{F}(s) \cdot \vec{E}_j) ds \quad (5.6)$$

and finally,

$$\vec{X}_p(t) = \mu_1 e^{\lambda_1 t} \vec{E}_1 \int_0^t e^{-\lambda_1 s} \vec{F}(s) \cdot \vec{E}_1 ds + \dots + \mu_n e^{\lambda_n t} \vec{E}_n \int_0^t e^{-\lambda_n s} \vec{F}(s) \cdot \vec{E}_n ds$$

We will illustrate with an example.

Consider the system

$$\vec{X}'(t) = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{X}(t) + \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix}$$

This coefficient matrix has the following eigenpairs

$$\vec{E}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \lambda_1 = -1, \quad \vec{E}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \lambda_2 = 2, \quad \vec{E}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \lambda_3 = 5,$$

Then (5.6) implies,

$$C_1(t) = \frac{-1}{2} \int_0^t e^s ds = \frac{1}{2} e^t - \frac{1}{2} t e^t - \frac{1}{2}$$

$$C_2(t) = \int_0^t e^{-2s} ds = \frac{1}{2} - \frac{1}{2} e^{-2t}$$

$$C_j(t) = \frac{1}{2} \int_0^t e^{-5s} ds = \frac{1}{50} - \frac{1}{10} t e^{-5t} - \frac{1}{50} e^{-5t}$$

and so the particular solution is

$$\vec{X}_p(t) = \left(\frac{1}{2} - \frac{1}{2} t - \frac{1}{2} e^{-t} \right) \vec{E}_1 + \left(\frac{1}{2} e^{2t} - \frac{1}{2} \right) \vec{E}_2 + \left(\frac{1}{50} e^{5t} - \frac{1}{10} t - \frac{1}{50} \right) \vec{E}_3$$

Exercises

Let $A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

1. Find $X_p(t)$ $\frac{d}{dt} \vec{X}(t) = A_1 \vec{X}(t) + \begin{bmatrix} 2t \\ -2 \end{bmatrix}$

2. Find $X_p(t)$ $\frac{d}{dt} \vec{X}(t) = A_2 \vec{X}(t) + \begin{bmatrix} t-1 \\ 0 \end{bmatrix}$

3. Find $X_p(t)$ $\frac{d}{dt} \vec{X}(t) = A_3 \vec{X}(t) + \begin{bmatrix} 1-t \\ 1 \\ 0 \end{bmatrix}$

4. Find $X_p(t)$ $\frac{d}{dt} \vec{X}(t) = A_4 \vec{X}(t) + \begin{bmatrix} 0 \\ t \\ e^{-t} \end{bmatrix}$